

# On Algebras That Almost Have Finite Dimensional Representations

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**Abstract.** We introduce the notion of almost finite dimensional representability of algebras and study its connection with the classical finiteness conditions.

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# 1 Introduction

In [8] Gordon and Vershik introduced the notion of *LEF*-groups. A group  $G$  is called *LEF* (locally embeddable into finite groups) if for any finite subset  $1 \in S \subset G$  there exists an injective map  $\psi_S : S \rightarrow G_S$  into a finite group  $G_S$  such that:  $\psi_S(1) = 1$  and  $\psi_S(ab) = \psi_S(a) \cdot \psi_S(b)$ , whenever  $a, b, ab \in S$ .

In the same paper they defined *LEF*-algebras as well. Let  $k$  be a commutative field and  $\mathcal{A}$  be  $k$ -algebra. Then  $\mathcal{A}$  is called *LEF* if for any finite dimensional subspace  $L \subset \mathcal{A}$  there exists an injective linear map  $\psi_L : L \rightarrow \text{End}_k(V)$  for some finite dimensional  $k$ -space  $V$ , such that:  $\psi_L(1) = 1$  and  $\psi_L(xy) = \psi_L(x)\psi_L(y)$ , whenever  $x, y, xy \in L$ .

Let us recall the three class of pathological rings from Cohn's classical paper [2].

- I There exist  $A \in \text{Mat}_{m \times n}(R)$  and  $B \in \text{Mat}_{n \times m}(R)$ ,  $m \neq n$ , such that  $AB = I$  and  $BA = I$  (in other words, the ring  $R$  does not have the *IBN property*).
- II There exist  $A \in \text{Mat}_{m \times n}(R)$ , and  $B \in \text{Mat}_{n \times m}(R)$ ,  $m > n$ , such that  $AB = I$  (or the ring  $R$  does not satisfy the *rank condition*).
- III There exist  $A \in \text{Mat}_{n \times n}(R)$ , and  $B \in \text{Mat}_{n \times n}(R)$ , such that  $AB = I$  and  $BA \neq I$  (or the ring  $R$  is not *stably finite*).

Obviously, stable finiteness implies the rank condition, and the rank condition in turn implies the IBN property. Cohn showed that this categories are strictly containing each other. Now, let us observe that *LEF*-algebras can not be pathological at all.

**Proposition 1.1** *LEF-algebras are stably finite.*

**Proof:** Let  $\mathcal{A}$  be a *LEF*-algebra. Suppose that  $A \in \text{Mat}_{n \times n}(\mathcal{A})$ ,  $B \in \text{Mat}_{n \times n}(\mathcal{A})$ , such that  $AB = I$ . Consider the finite dimensional subspace  $L$  generated by  $1 \in \mathcal{A}$ , the entries of  $A$  and  $B$  and all the possible products in the form of  $xy$ , where  $x$  is an entry of  $A$  and  $y$  is an entry of  $B$ . Let us consider an injective map  $\psi_L : L \rightarrow \text{End}_k(V)$  as in the definition of *LEF*-algebras. Obviously it extends to a map  $\widehat{\psi}_L : \text{Mat}_{n \times n}(L) \rightarrow \text{End}_k(V^n)$  such that  $\widehat{\psi}_L(A)\widehat{\psi}_L(B) = I \in \text{End}_k(V^n)$  and  $\widehat{\psi}_L(B)\widehat{\psi}_L(A) \neq I$ , where  $\text{Mat}_{n \times n}(L)$  is the finite dimensional vector space of matrices with entries from  $L$ . The contradiction is clear, since the finite dimensional matrix algebras are stably finite. ■

There is a rather simple ring theoretical obstruction as well that excludes, say, all the Weyl-algebras from the category of *LEF*-algebras.

**Proposition 1.2** *Let  $\mathcal{A}$  be a finitely presented infinite dimensional simple affine algebra. Then  $\mathcal{A}$  is not a *LEF*-algebra.*

**Proof:** Let  $a_1, a_2, \dots, a_n$  be generators of  $\mathcal{A}$  and let  $f_i(a_1, a_2, \dots, a_n) = 0$  be presentations,  $1 \leq i \leq l$ , where the  $f_i$ 's are polynomials of non-commuting variables. Easy to see that there exists a finite dimensional subspace  $L \subset \mathcal{A}$  and an injective linear map  $\psi_L : L \rightarrow \text{End}_k(V)$  into a finite dimensional matrix algebra over  $k$  such that;

$$f_i(\psi_L(a_1), \psi_L(a_2), \dots, \psi_L(a_n)) = 0, \quad 1 \leq i \leq l.$$

Therefore the algebra generated by the matrices  $\psi_L(a_1), \psi_L(a_2), \dots, \psi_L(a_n)$  is a non-trivial quotient of  $\mathcal{A}$ , this leads to a contradiction. ■

In [6], Gromov defined a weaker version of the *LEF*-property for groups, that he called initial amenability. This led us to the following definition.

**Definition 1.1** *A unital  $k$ -algebra  $\mathcal{A}$  almost has finite dimensional representations if for any finite dimensional subspace  $1 \in L \subset \mathcal{A}$  and  $\epsilon > 0$ , there exists a finite dimensional vector space  $V$  together with a subspace  $V_\epsilon \subset V$  such that*

- *There exists a linear (not necessarily injective) map  $\psi_{L,\epsilon} : L \rightarrow \text{End}_k(V)$  such that  $\psi_{L,\epsilon}(1) = I$  and  $\psi_{L,\epsilon}(a)\psi_{L,\epsilon}(b)(v) = \psi_{L,\epsilon}(ab)(v)$  whenever  $a, b, ab \in L$  and  $v \in V_\epsilon$ .*
- $\frac{\dim_k V - \dim_k V_\epsilon}{\dim_k V} < \epsilon$ .

We shall call such maps  $\epsilon$ -almost representations of  $L$ . Obviously algebras (e.g. group algebras) that actually have finite dimensional representations or *LEF*-algebras are in the class above. Clearly, if  $\mathcal{A}$  almost has finite dimensional representations so does any subalgebra  $\mathcal{B} \subset \mathcal{A}$  containing the unit of  $\mathcal{A}$ . Also, if  $\mathcal{A}$  almost has finite dimensional representations and  $\mathcal{A}$  is a quotient of the algebra  $\mathcal{B}$ , then  $\mathcal{B}$  almost has finite dimensional representations. Note that we shall give examples of simple infinite dimensional algebras almost having finite dimensional representations. Our main results are the following two theorems.

### Theorem 1

1. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $k$ -algebras almost having finite dimensional representation, then both  $\text{Mat}_{n \times n} \mathcal{A}$  and  $\mathcal{A} \otimes_k \mathcal{B}$  are algebras almost having finite dimensional representations.*
2. *If  $\mathcal{A}$  almost has finite dimensional representations then it satisfies the rank condition.*

**Theorem 2** *If  $\mathcal{A}$  is a simple algebra almost having finite dimensional representations then it is stably finite.*

Actually, we shall construct an ideal  $RR(\mathcal{A})$  in any  $k$ -algebra  $\mathcal{A}$  such that  $RR(\mathcal{A}) \neq \mathcal{A}$  if and only if  $\mathcal{A}$  almost has a finite dimensional representation and if  $RR(\mathcal{A}) = 0$  then  $\mathcal{A}$  is stably finite. For any pair of algebras  $\mathcal{A}, \mathcal{B}$  and any algebra homomorphism  $\tau : \mathcal{A} \rightarrow \mathcal{B}$ , we shall see that  $\tau(RR(\mathcal{A})) \subset RR(\mathcal{B})$ .

## 2 Examples

In [7], Rowen proved that any algebra of subexponential growth is an IBN-algebra. In [5], we extended Rowen's result to the larger class of *amenable algebras*.

**Definition 2.1** *The  $k$ -algebra  $\mathcal{A}$  is amenable if for any finite dimensional linear subspace  $1 \in B \subset \mathcal{A}$  and  $\epsilon > 0$  there exists a non-trivial finite dimensional linear subspace  $Q \subset \mathcal{A}$  such that*

$$\frac{\dim_k BQ - \dim_k Q}{\dim_k Q} < \epsilon.$$

Here  $BQ$  denotes the subspace spanned by the vectors in the form  $bq$ , where  $b \in B$  and  $q \in Q$ .

**Proposition 2.1** *If  $\mathcal{A}$  is amenable, then it almost has finite dimensional representations.*

**Proof:** Fix a finite dimensional subspace  $1 \in L \subset \mathcal{A}$ . Let  $\{Q_n\}_{n \geq 1}$  be a sequence of finite dimensional subspaces in  $\mathcal{A}$  such that

$$\frac{\dim_k LQ_n - \dim_k Q_n}{\dim_k Q_n} < \frac{1}{n}$$

and consider linear subspaces  $T_n$  such that  $Q_n \oplus T_n = \mathcal{A}$  as vector spaces. Denote the projection onto  $Q_n$  by  $P_n$ . Now, let  $\psi_n : L \rightarrow \text{End}_k(Q_n)$  be defined as

$$\psi_n(a)v = P_n(av).$$

Note that  $\psi_n(ab)(v) = \psi_n(a)\psi_n(b)(v)$  if  $P_n(bv) = bv$ , that is  $v \in \text{Ker}(m_b - P_n m_b)$  where  $m_b$  is the left-multiplication by the element  $b$ . Observe that  $\dim_k \text{Ran}(m_b - P_n m_b) \leq \frac{1}{n} \dim_k Q_n$ . Hence

$$\dim_k \text{Ker}(m_b - P_n m_b) \geq \frac{n-1}{n} \dim_k Q_n.$$

Consequently, for any  $0 < \epsilon < 1$ ,

$$\dim_k \bigcap_{x \in L} \text{Ker}(m_x - P_n m_x) \geq (1 - \epsilon) \dim_k Q_n,$$

if  $n$  is large enough. Let  $Q_n^\epsilon = \bigcap_{x \in L} \text{Ker}(m_x - P_n m_x)$ . Then, if  $v \in Q_n^\epsilon$ :

$$\psi_n(xy)(v) = \psi_n(x)\psi_n(y)(v),$$

provided that  $x, y, xy \in L$ . Also,

$$\frac{\dim_k Q_n - \dim_k Q_n^\epsilon}{\dim_k Q_n} < \epsilon.$$

Hence  $\mathcal{A}$  almost has finite dimensional representations. ■

Note that by Theorem 1. we can immediately see that amenable algebras satisfy the rank condition.

There is however an even more direct relation between amenability and the property studied in our paper. Let  $G(V, E)$  be an infinite connected graph of bounded vertex degree. Then  $V(G)$  is a discrete metric space with the shortest path-metric  $d_G$ . The graph is called *amenable* if its isoperimetric constant is zero, or in other words, there exists finite subsets  $F_n \subset V(G)$  such that for any  $k \geq 1$ :

$$\lim_{n \rightarrow \infty} \frac{|B_k(F_n)|}{|F_n|} = 1,$$

where  $B_k(F_n)$  is the  $k$ -neighbourhood of  $F_n$  in the metric  $d_G$ :

$$B_k(F_n) = \{y \in V : \text{there exists } x \in F_n; d_G(x, y) \leq k\}.$$

The translation algebra  $T_k(G)$  of  $G$  is the set of all square matrices  $A$  indexed by  $V \times V$  with entries from  $k$  such that  $A(x, y) = 0$ , whenever  $d(x, y) > A_l$  for some constant depending only  $A$  [1]. The following result is motivated by Proposition 4.1 [1].

**Proposition 2.2** *The graph  $G$  is amenable if and only if the translation algebra  $T_k(G)$  almost has finite dimensional representations.*

**Proof:** Suppose that  $G$  is amenable. Let  $Q_n$  be the vectorspace over  $k$  spanned by the elements of  $F_n$  as a formal basis. Then for any finite dimensional subspace  $1 \in L \subset T_k(G)$ , clearly

$$\lim_{n \rightarrow 0} \frac{\dim_k LQ_n - \dim_k Q_n}{\dim_k Q_n} = 0.$$

Thus one can use exactly the same argument as in Proposition 2.1 to show that  $T_k(G)$  almost has finite dimensional representations. If  $G$  is non-amenable, then by [3] (see also [4]) there exists a partition  $V = V_1 \cup V_2$  and bijective maps  $\phi_1 : V \rightarrow V_1$ ,  $\phi_2 : V \rightarrow V_2$  such that

$$\sup_{x \in V} d_G(x, \phi_1(x)) < \infty, \quad \sup_{x \in V} d_G(x, \phi_2(x)) < \infty \quad (1)$$

Let  $A(x, y) = 1$  if  $x = \psi_1(y)$ , otherwise let  $A(x, y) = 0$ . Similarly, let  $B(x, y) = 1$  if  $x = \psi_2(y)$ , otherwise let  $B(x, y) = 0$ . By (1) the matrices  $A$  and  $B$  are in  $T_k(G)$ . Obviously,  $AA^T = I$ ,  $BB^T = I$  and  $A^T A + B^T B = I$ . Therefore the algebra generated by  $A, A^T, B, B^T$  is not an IBN-algebra, hence  $T_k(G)$  is not an IBN-algebra as well. By Theorem 1. the proposition follows. ■

### 3 The proof of Theorem 1.

Suppose that  $\mathcal{A}$  almost has finite dimensional representations. Let  $L \subset \text{Mat}_{n \times n}(\mathcal{A})$  be a finite dimensional  $k$ -space. Then there exists a finite dimensional  $k$ -space  $M \subset \mathcal{A}$  such that  $L \subset \text{Mat}_{n \times n}(M)$ . Let  $\psi_{M, \epsilon} : M \rightarrow \text{End}_k(V)$  be an  $\epsilon$ -almost representation of  $M$ . Then we can extend it to  $\widehat{\psi}_{M, \epsilon} : \text{Mat}_{n \times n}(M) \rightarrow \text{End}_k(V^n)$  in a natural manner. If  $s \in (V_\epsilon)^n$ , then  $\widehat{\psi}_{M, \epsilon}(ab)(s) = \widehat{\psi}_{M, \epsilon}(a)\widehat{\psi}_{M, \epsilon}(b)(s)$ . Obviously,  $\lim_{\epsilon \rightarrow 0} \frac{\dim_k V^n - \dim_k (V_\epsilon)^n}{\dim_k V^n} = 0$ . Hence  $\text{Mat}_{n \times n}(\mathcal{A})$  almost has finite dimensional representations.

Now let  $\mathcal{A}$  and  $\mathcal{B}$  be algebras almost having finite dimensional representations. If  $L \subset \mathcal{A} \otimes_k \mathcal{B}$  is a finite dimensional vectorspace, then  $L \subset M \otimes_k N$ , where  $M \subset \mathcal{A}$  and  $N \subset \mathcal{B}$  are finite dimensional vectorspaces. If we choose  $\epsilon$ -almost representations  $\psi_{M, \epsilon} : M \rightarrow \text{End}_k(V)$ ,  $\psi_{N, \epsilon} : N \rightarrow \text{End}_k(W)$ , then we obtain a linear map

$$\psi_{M, \epsilon} \otimes \psi_{N, \epsilon} : M \otimes N \rightarrow \text{End}_k(V) \otimes \text{End}_k(W) \sim \text{End}_k(V \oplus W).$$

Obviously,

$$(\psi_{M, \epsilon} \otimes \psi_{N, \epsilon})(ac \otimes bd)(v, w) = (\psi_{M, \epsilon} \otimes \psi_{N, \epsilon})(a \otimes b) \cdot (\psi_{M, \epsilon} \otimes \psi_{N, \epsilon})(c \otimes d)(v, w),$$

provided that  $v \in V_\epsilon$  and  $w \in W_\epsilon$ . Consequently,  $\mathcal{A} \otimes_k \mathcal{B}$  almost has finite dimensional representations.

Finally, suppose that  $A \in \text{Mat}_{m \times n}(\mathcal{A})$ ,  $B \in \text{Mat}_{n \times m}(\mathcal{A})$ ,  $m > n$  and  $AB = I$ . Then, for any  $\epsilon$  we have finite dimensional  $k$ -spaces  $V$ ,  $V_\epsilon$ ,  $\frac{\dim_k V - \dim_k V_\epsilon}{\dim_k V} < \epsilon$  and linear maps

$A_\epsilon : V^n \rightarrow V^m$ ,  $B_\epsilon : V^m \rightarrow V^n$  such that  $A_\epsilon B_\epsilon(v) = v$ , if  $v \in V_\epsilon^m$ . Note however that the range of  $A_\epsilon B_\epsilon$  is at most  $n \cdot \dim_k V$ - dimensional. However, if  $\epsilon$  is small enough,

$$m \cdot \dim_k V_\epsilon > n \cdot \dim_k V ,$$

leading to a contradiction. ■

## 4 The rank radical

In order to prove Theorem 2. we need the notion of a rank radical of an algebra,  $RR(\mathcal{A})$ .

**Definition 4.1** *If  $\mathcal{A}$  is not an algebra that almost has a finite dimensional representation, set  $RR(\mathcal{A}) = \mathcal{A}$ . Otherwise, let  $p \in RR(\mathcal{A})$  if there exists a finite dimensional subspace  $\{1, p\} \subset L \subset \mathcal{A}$  such that for any  $\delta > 0$  there exists  $n_\delta > 0$  with the following property: If  $0 < \epsilon < n_\delta$  and  $\psi_{L,\epsilon} : L \rightarrow \text{End}_k(V)$  is an  $\epsilon$ -almost representation then*

$$\dim_k \text{Ran} (\psi_{L,\epsilon}(p)) < \delta \dim_k V .$$

*Note that if  $\phi : \mathcal{A} \rightarrow \text{End}_k(V)$  is an algebraic homomorphism then  $\phi(RR(\mathcal{A})) = 0$ .*

**Proposition 4.1**  *$RR(\mathcal{A})$  is an ideal.*

Let  $a \in \mathcal{A}$ ,  $p \in RR(\mathcal{A})$  be arbitrary elements and  $\{a, p\} \subset L \subset \mathcal{A}$  be the finite dimensional vectorspace of the previous definition. Let  $L \subset L' \subset \mathcal{A}$  be any finite dimensional vectorspace that contains  $a$  and  $ap$ . If  $\epsilon < n_{\frac{1}{2}\delta}$ :

$$\dim_k \text{Ran} (\psi_{L',\epsilon}(p)) \leq \frac{1}{2}\delta \cdot \dim_k V$$

since  $L \subset L'$ . Note that  $\psi_{L',\epsilon}(ap)(v) = \psi_{L',\epsilon}(a) \cdot \psi_{L',\epsilon}(p)(v)$  if  $v \in V_\epsilon$ , thus

$$\dim_k \text{Ran} (\psi_{L',\epsilon}(ap)) \leq \frac{1}{2}\delta \cdot \dim_k V + \epsilon \cdot \dim_k V .$$

Therefore if  $\epsilon$  is small enough then

$$\dim_k \text{Ran} (\psi_{L',\epsilon}(ap)) \leq \delta \cdot \dim_k V .$$

Hence  $ap \in RR(\mathcal{A})$ . Similarly,  $pa \in RR(\mathcal{A})$ . ■

Observe that if  $p \notin RR(\mathcal{A})$ , then for any finite dimensional subspace  $\{1, p\} \subset L \subset \mathcal{A}$  and  $\epsilon > 0$  there exists an  $\epsilon$ -almost representation such that

$$\dim_k \text{Ran} (\psi_{L,\epsilon}(p)) \geq C_L \cdot \dim_k V ,$$

where  $C_L$  depends only on  $L$ . Hence the following proposition easily follows :

**Proposition 4.2** *Let  $\mathcal{A}, \mathcal{B}$  be  $k$ -algebras and let  $\tau : \mathcal{A} \rightarrow \mathcal{B}$  be an algebra homomorphism. Then  $\tau(RR(\mathcal{A})) \subset RR(\mathcal{B})$ .*

Now we prove a statement that immediately implies Theorem 2. by Proposition 4.1.

**Proposition 4.3** *Suppose that  $RR(\mathcal{A}) = 0$ , then  $\mathcal{A}$  is stably finite.*

**Proof:** Let  $T, S \in \text{End}_k(k^l)$ , suppose that  $V \subseteq k^l$ ,  $\dim_k(V) \geq (1 - \epsilon)l$  and  $TS(v) = v$  if  $v \in V$ . Then

$$\dim_k \text{Ran}(TS - ST) \leq 2\epsilon l. \quad (2)$$

Indeed, if  $w \in S(V)$ , then  $ST(w) = w$ . Hence if  $w \in V \cap S(V)$ , then  $(TS - ST)v = 0$ . Obviously,  $\dim_k(V \cap S(V)) \geq (1 - 2\epsilon)l$ . Thus (2) follows. Now suppose that  $A \in \text{Mat}_{n \times n}(\mathcal{A})$ ,  $B \in \text{Mat}_{n \times n}(\mathcal{A})$ ,  $AB = I$ ,  $BA = P \neq I$ . Let  $L$  be a finite dimensional vectorspace spanned by the elements in the form  $1, x, y, xy, yx$ , where  $x$  is an entry of  $A$  and  $y$  is an entry of  $B$ . Let  $p \in L$  be a non-zero entry of  $AB - BA$ .

Let  $\{\psi_L^m : L \rightarrow \text{End}_k(V_m)\}_{m \geq 1}$  be a sequence of  $\frac{1}{m}$ -almost representations such that

$$\dim_k \text{Ran}(\psi_L^m(p)) > C \cdot \dim_k V_m \quad (3)$$

for some constant  $C > 0$ . We have the extension:

$$\widehat{\psi}_L^m : \text{Mat}_{n \times n}(L) \rightarrow \text{End}_k((V_m)^n)$$

such that  $\widehat{\psi}_L^m(A) \cdot \widehat{\psi}_L^m(B)(v) = v$  if  $v \in W_m \subset (V_m)^n$ , where

$$\lim_{m \rightarrow \infty} \frac{n \cdot \dim_k V_m - \dim_k W_m}{n \cdot \dim_k V_m} = 0.$$

By (3)

$$\frac{\dim_k \text{Ran}(\widehat{\psi}_L^m(AB - BA))}{n \cdot \dim_k V_m}$$

can not converge to zero. On the other hand, by the observation at the beginning of the proof

$$\lim_{m \rightarrow \infty} \frac{\dim_k \text{Ran}(\widehat{\psi}_L^m(A)\widehat{\psi}_L^m(B) - \widehat{\psi}_L^m(B)\widehat{\psi}_L^m(A))}{n \cdot \dim_k V_m} = 0$$

in contradiction with the fact that

$$\widehat{\psi}_L^m(AB - BA)(v) = (\widehat{\psi}_L^m(A)\widehat{\psi}_L^m(B) - \widehat{\psi}_L^m(B)\widehat{\psi}_L^m(A))(v)$$

if  $v \in W_m$ . ■

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